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STEKLOV INEQUALITY AND ITS APPLICATION

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Abstract. In this paper, one lower estimate for the first eigenvalue of the Laplace operator is obtained. To this end, a Steklov-type inequality for bounded and unbounded domains with a finite measure in proved based on the Sobolev inequality.

Keywords: first eigenvalue, Laplace operator, lower bound, Sobolev inequality, Steklov inequality, lack of eigenfunctions.

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1. Introduction

The issue of calculation of the lower estimate for the first eigenvalue of the Laplace operator is important. This problem was studied by different authors. Finding the best estimate for the first eigenvalue of the Laplace operator is studied in scientific literature. The proof of a Steklov-type inequality for bounded and unbounded domains with a finite measure is also important since the Steklov inequality is widely used in various sections of mathematical physics. Finding a lower estimate for the first eigenvalue of the Laplace operator is relevant, since this question arises in applied problems. In this note, an attempt is made to find a lower estimate for the first eigenvalue of the Laplace operator. For this purpose, Steklov-type inequality in bounded and unbounded domains with a finite measure is proved on the basis of the Sobolev inequality by passing to the limit. The study of the relationship between the Steklov and Sobolev inequalities is relevant and of scientific interest. In this paper, the Steklov inequality is applied to calculation of the lower estimate for the first eigenvalue of the Laplace operator.

2. Denotation and formulation of basic results

Let $\Omega \subset R^n$ be the a bounded or unbounded domain with a finite measure $|\Omega|$. For convenience of the further statement we accept the following notations $\|u\|_p = \left\{ \int\limits_\Omega |u(x)|^p \, dx \right\}^{1/p}$, $p \geq 1$ is a norm in $L_p(\Omega)$; the index p in $\|.\|_p$ will be omitted for $p = 2$, i.e. we will write $\|\cdot\|$.

Let $|\Omega|$ is be a bounded domain with a rather smooth boundary. For the Laplace operator Δ, we consider the following spectral problem:

$$\begin{aligned} &\Delta u + \lambda u = 0, \text{ in } \Omega, \, \lambda > 0, \\ &u = 0 \text{ on } \partial\Omega. \end{aligned} \tag{1}$$

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As it is known (see for instance [1, p. 434]), problem (1) has a nontrivial solution both in classical and in the generalized sense only for discrete set of positive values $\{\lambda_k\}$ of the parameter λ such that of $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ... \leq \lambda_{k.....}\lambda_k \to \infty$ as $k \to \infty$.

The following theorems are valid.

Theorem 2.1. *Let λ_1 be the first eigenvalue of problem* (1)*. Then the following lower bound is valid for it:*

$$\lambda_1 \geq \frac{\gamma_1}{|\Omega|^{2/n}}, \tag{2}$$

where $\gamma_1 = \frac{\pi e n}{2}$.

Theorem 2.2. *Let λ in problem* (1) *satisfy the condition*

$$\lambda < \frac{\gamma_1}{|\Omega|^{2/n}}, \tag{3}$$

where λ_1 was determined in (2)*. Then spectral problem* (1) *has no trivial solutions, i.e. eigenfunctions in the class* $\overset{0}{W}{}_2^1(\Omega)$.

Theorem 2.1 follows from the following conjectures.

Conjecture 2.1. *Let Ω be a bounded or unbounded domain with a finite measure $|\Omega|$. Then, for any function from the class $\overset{0}{W}{}_2^1(\Omega)$ the following inequality is valid:*

$$\|u\| \leq w_1 \|\nabla u\|, \tag{4}$$

where $w_1 = \sqrt{\frac{2}{\pi n e}\,|\Omega|^{\frac{2}{n}}}$.

Conjecture 2.2. *Let λ_1 be the first eigenvalue of problem* (1)*. Then, for $\forall u(x) \in \overset{0}{W}{}_2^1$ the following exact inequality is valid:*

$$\|u\| \leq w_2 \|\nabla u\|, \tag{5}$$

where $w_2 = \frac{1}{\sqrt{\lambda_1}}$. These inequality was first proved by Steklov in 1896 [2], see also [3. p.33].

Now theorem 2.1 follows from conjecture 2.1 by virtue of conjecture 2.2.
In [4], the following estimate is proved for the first eigenvalue of problem (1) for $n = 2$:

$$\lambda_1 \geq \frac{\pi}{|\Omega|}. \tag{6}$$

In the paper [5], for special domains ("plane-covering domain") the estimate

$$\lambda_1 \geq \frac{2\pi}{\Omega} \tag{7}$$

is proved.

Comparison of estimates (2), (6) and (7) shows that our estimate is the best one.

3. Sobolev inequality

We give the proof of Conjecture 2.1.

Introduce the following denotation. For the given ρ from the interval $(0,\, \rho_0)$, where $\rho_0 = \infty$ for $n = 1,\, 2$; $\rho_0 = 4/(n-2)$ for $n \geq 3$ we define $\alpha = \rho n / \left[2\left(\rho + 2\right)\right]$. For the given $\alpha \in (0,1)$, we define $\chi = \sqrt{\alpha^{\alpha}\left(1-\alpha\right)^{1-\alpha}}$. Let $B = B\left(n/2,\, n\left(1-\alpha\right)/2\alpha\right)$ be the Euler beta function, $\Gamma\left(n/2\right)$ be the Euler gamma-function. We put

$$k\left(\alpha\right) = \frac{\left(\sigma_n B/2\right)^{\alpha/n}}{\chi}, \tag{8}$$

where $\sigma_n = 2\pi^{n/2}\Gamma\left(n/2\right)$.

For the given $p_0 \in (1,\, 2)$, we define

$$k_B\left(p_0\right) = \left[\frac{\left(p_0/2\pi\right)^{1/p_0}}{\left(p_0^1/2\pi\right)^{1/p_0^1}}\right]^{n/2}, \tag{9}$$

where $\frac{1}{p_0} + \frac{1}{p_0^1} = 1$.
We introduce two denotations:

$$\overline{k_0\left(\alpha\right)} = k\left(\alpha\right) k_B\left(\frac{\rho+2}{\rho+1}\right) = k\left(\alpha\right) k_B\left(\frac{2n}{n+2\alpha}\right). \tag{10}$$

$$\overline{k_C}\left(\Omega,\, \alpha\right) = \overline{k_0}^{1/\alpha}\left(\alpha\right)\left|\Omega\right|^{\frac{1-\alpha}{n}} \quad .\ \left|\Omega\right| = mes\Omega. \tag{11}$$

Conjecture 3.1. *Let Ω be a bounded or unbounded domain with a finite measure $\left|\Omega\right|$, ρ be a number determined above, $u\left(x\right)$ be any function from the class $\overset{0}{W}{}_2^1\left(\Omega\right)$. Then, the following Sobolev inequality is valid:*

$$\left\|u\right\|_{\rho+2} \leq \overline{k_C}\left(\Omega,\, \alpha\right)\left\|\nabla u\right\|, \tag{12}$$

where $\overline{k}_C$ was determined by formula (11), $\overline{k}_0\left(\alpha\right)$, k_B, $k\left(\alpha\right)$ by formulas (10), (9), (8) respectively.

Proof. Conjecture 3.1 is proved based on the results obtained [6] (see also [7]). □

The following lemma is valid:

Lemma 3.1. *Let ρ and α be the numbers determined above, $\overline{k}_0\left(\alpha\right)$ be determined by formula (10) k_B from (9), $k\left(\alpha\right)$ from (8). Then, let $u\left(x\right)$ be any function from the class $W_2^1\left(R^n\right)$. Then the imbedding $W_2^1\left(R^n\right) \subset L_{\rho+2}\left(R^n\right)$ and interpolation Sobolev inequality*

$$\left\|u\right\|_{\rho+2} \leq \overline{k_0}\left\|\nabla u\right\|^{\alpha}\left\|u\right\|^{1-\alpha} \tag{13}$$

is valid, here $\left\|.\right\|_{\rho+2}$ is a norm in $L_{\rho+2}\left(R^n\right)$ $\left\|\cdot\right\|$ is a norm in $L_2\left(R^n\right)$.

Apply inequality (13) to the function $u\left(x\right) \in \overset{0}{W}{}_2^1\left(\Omega\right)$ and estimate the norm $\left\|u\right\|$ by $\left\|u\right\|_{\rho+2}$, applying the Holder inequality:

$$\int\limits_{\Omega}\left|u\left(x\right)\right|^2 dx \leq \left(\int\limits_{\Omega}\left|u\left(x\right)\right|^{\rho+2} dx\right)^{\frac{2}{\rho+2}}\left(\int\limits_{\Omega} dx\right)$$

or

$$\left\|u\right\| \leq \left\|u\right\|_{\rho+2}\left|\Omega\right|^{\frac{\rho}{2(\rho+2)}}. \tag{14}$$

From $\left(\int\limits_{\Omega}|u\,(x)|^{\rho+2}\,dx\right)^{\frac{1}{\rho+2}} \leq \overline{k}_0\,(\alpha)\left(\int\limits_{\Omega}|\nabla u\,(x)|^2\,dx\right)^{\frac{\alpha}{2}}\left(\int\limits_{\Omega}|u(x)|^2\,dx\right)^{\frac{1-\alpha}{2}}$, by virtue of (14) we get (12):

$$\|u\|_{\rho+2} \leq \left[\overline{k_0}\,(\alpha)\right]^{\frac{1}{\alpha}}|\Omega|^{\frac{1-\alpha}{n}}\|\nabla u\| = \overline{k}_C\,(\Omega,\,\alpha)\,\|\nabla u\|\,.$$

4. Limit passage

In (12), we pass to the limit as $\rho \to 0+\ (\alpha \to 0+)$, where $\overline{k}_0 = k\,(\alpha)\,k_B\left(\frac{2n}{n+2\alpha}\right)$. Represent $k_B\left(\frac{2n}{n+2\alpha}\right)^{\frac{1}{\alpha}}$ in the following form:

$$\left[k_B\left(\frac{2n}{n+2\alpha}\right)\right]^{\frac{1}{\alpha}} = \frac{1}{\pi\sqrt{\left(1+\frac{2\alpha}{n}\right)^{\frac{n}{2\alpha}}}\sqrt{\left(1+\frac{2\alpha}{n}\right)\left(1-\frac{2\alpha}{n}\right)^{\frac{-n}{2\alpha}}}\sqrt{1-\frac{2\alpha}{n}}}. \tag{15}$$

Passing to the limit in (15) as $\alpha \to 0+$, we get:

$$\lim_{\alpha\to 0+}\left[k_B\left(\frac{2n}{n+2\alpha}\right)\right]^{\frac{1}{\alpha}} = \frac{1}{\pi e}. \tag{16}$$

Using the relation $B\left(\frac{n}{2},\,\frac{n(1-\alpha)}{2\alpha}\right) = \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n(1-\alpha)}{2\alpha}\right)}{\Gamma\left(\frac{n}{2\alpha}\right)}$, we represent $[k\,(\alpha)]^{\frac{1}{\alpha}}$ in the following form:

$$[k\,(\alpha)]^{\frac{1}{\alpha}} = \sqrt{\pi}\sqrt{1-\alpha}\sqrt{(1-\alpha)^{-\frac{1}{\alpha}}}\left[\frac{\Gamma\left(\frac{n(1-\alpha)}{2\alpha}\right)}{\Gamma\left(\frac{n}{2\alpha}\right)}\right]^{\frac{1}{n}}\frac{1}{\sqrt{\alpha}}. \tag{17}$$

Calculate $\lim\limits_{\alpha\to 0+}[k\,(\alpha)]^{\frac{1}{\alpha}}$ obviously, $\lim\limits_{\alpha\to 0+}\sqrt{1-\alpha}\sqrt{(1-\alpha)^{-\frac{1}{\alpha}}} = \sqrt{e}$.
Calculate

$$\lim_{\alpha\to 0+}\left[\frac{\Gamma\left(\frac{n(1-\alpha)}{2\alpha}\right)}{\Gamma\left(\frac{n}{2\alpha}\right)}\right]^{\frac{1}{n}}\frac{1}{\sqrt{\alpha}}. \tag{18}$$

To this end, we use the following relation for the gamma-function $\Gamma\,(\theta)$ [8]:

$$\ln\Gamma\,(\theta) = \theta\ln\theta - \theta - \frac{1}{2}\ln\theta + \frac{1}{2}\ln 2\pi + R_0\,(\theta)\,, \tag{19}$$

where $R_0\,(\alpha) = \frac{1}{2}\sum\limits_{m=1}^{\infty}\frac{m}{(m+1)(m+2)}\sum\limits_{j=1}^{\infty}\frac{1}{(\theta+j)^{m+1}}$. For $h\,(\alpha) = \left[\frac{\Gamma\left(\frac{n(1-\alpha)}{2\alpha}\right)}{\Gamma\left(\frac{n}{2\alpha}\right)}\right]^{\frac{1}{n}}\frac{1}{\sqrt{\alpha}}$ we have

$$\ln h\,(\alpha) = \frac{1}{n}\ln\left[\Gamma\frac{(n\,(1-\alpha))}{2\alpha}\right] - \frac{1}{n}\ln\left[\Gamma\left(\frac{n}{2\alpha}\right)\right] - \frac{1}{2}\ln\alpha. \tag{20}$$

By formula (19) for $\ln\left[\Gamma\left(\frac{n(1-\alpha)}{2\alpha}\right)\right]$ and $\ln\left[\Gamma\left(\frac{n}{2\alpha}\right)\right]$ the following relations are valid:

$$\begin{gathered}\ln\left[\Gamma\left(\frac{n(1-\alpha)}{2\alpha}\right)\right] = \tfrac{n-n\alpha}{2\alpha}\ln\left(\tfrac{n-n\alpha}{2\alpha}\right) - \\ -\tfrac{n-n\alpha}{2\alpha} - \tfrac{1}{2}\ln\tfrac{n-\alpha n}{2\alpha} + \tfrac{1}{2}2\pi + R_1\,(\alpha)\,,\end{gathered} \tag{21}$$

where

$$R_1(\alpha) = \frac{1}{2}\sum_{m=1}^{\infty}\frac{m}{(m+1)(m+2)}\sum_{j=1}^{\infty}\frac{(2\alpha)^{m+1}}{(n+n\alpha+2\alpha j)^{m+1}}. \tag{22}$$

$$\ln\Gamma\left(\frac{n}{2\alpha}\right) = \frac{n}{2\alpha}\ln\frac{n}{2\alpha} - \frac{n}{2\alpha} - \frac{1}{2}\ln\frac{n}{2\alpha} + \frac{1}{2}\ln 2\pi + R_2(\alpha),$$

where $R_2(\alpha) = \frac{1}{2}\sum\limits_{m=1}^{\infty}\frac{m}{(m+1)(m+2)}\sum\limits_{j=1}^{\infty}\frac{(2\alpha)^{m+1}}{(n+2\alpha j)^{m+1}}$.

Taking into account relations (21), (22) in (20), form $\ln h(\alpha)$ we have:

$$\ln h(\alpha) = -\frac{1}{2}\ln\frac{2}{n} + \frac{1-\alpha\ln(1-\alpha)}{\alpha} - \frac{1}{2n}\ln(1-\alpha) + \frac{1}{2} + \frac{1}{n}R_1(\alpha) - \frac{1}{n}R_2(\alpha).$$

Hence as $\alpha \to 0+$ we get $\lim\limits_{\alpha\to 0+}\ln h(\alpha) = \frac{1}{2}\ln\frac{2}{n}$ or

$$\lim_{\alpha\to 0+} h(\alpha) = \sqrt{\frac{2}{n}}. \tag{23}$$

From (17), (18), (23) it follows that

$$\lim_{\alpha\to 0+}[k_0(\alpha)]^{1/\alpha} = \sqrt{\frac{2\pi e}{n}}. \tag{24}$$

Finally, from (16), (24) we have:

$$\lim_{\alpha\to 0+}[\overline{k}_0(\alpha)]^{1/\alpha} = \sqrt{\frac{2}{n\pi e}}$$

So, in inequality (12), passing to the limit as $\alpha \to 0+$, we get formula (4).

5. Proof of theorem 2.2

We multiply the equation $\Delta u + \lambda u = 0$ by $u(x)$ and integrate the obtained relation with respect to Ω with regard to the boundary condition $u/\partial\Omega = 0$ and as a result we get $-\|\nabla u\|^2 + \lambda\|u\|^2 = 0$. Hence, by virtue of the inequality

$$\|u\| \le \sqrt{\frac{2}{n\pi e}}\,|\Omega|^{\frac{1}{n}}\,\|\nabla u\|$$

, we get the following inequality :

$$\left(\frac{n\pi e}{2|\Omega|^{\frac{2}{n}}} - \lambda\right)\|u\|^2 \le 0. \tag{25}$$

From (25), provided $\lambda < \frac{\pi n e}{2|\Omega|^{\frac{2}{n}}}$ it follows that $\|u\| = 0$, i.e. $u(x) \equiv 0$ Q.E.D. Theorem 2.2 is proved.

6. Conclusions

In this paper, an attempt is made to estimate from below the first eigenvalue of the Laplace operator that arises in applied mathematics, and the resulting estimate may be useful for applied scientists.

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